

Linear ROD subsets of Borel partial orders are countably cofinal in Solovay's model

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Abstract

The following is true in the Solovay model.

1. If $\langle D; \leq \rangle$ is a Borel partial order on a set D of the reals, $X \subseteq D$ is a ROD set, and $\leq \restriction X$ is linear, then $\leq \restriction X$ is countably cofinal.
2. If in addition every countable set $Y \subseteq D$ has a strict upper bound in $\langle D; \leq \rangle$ then the ordering $\langle D; \leq \rangle$ has no maximal chains that are ROD sets.

Linear orders, which typically appear in conventional mathematics, are countably cofinal. In fact *any* Borel (as a set of pairs) linear order on a subset of a Polish space is countably cofinal: see, e.g., [1]. On the other hand, there is an uncountably-cofinal quasi-order of class Σ_1^1 on $\mathbb{N}^{\mathbb{N}}$.

Example 1. Fix any recursive enumeration $Q = \{q_k : k \in \mathbb{N}\}$ of the rationals. For any ordinal $\xi < \omega_1$, let X_ξ be the set of all points $x \in \mathbb{N}^{\mathbb{N}}$ such that the maximal well-ordered (in the sense of the usual order of the rationals) initial segment of the set $Q_x = \{q_k : x(k) = 0\}$ has the order type ξ . Thus $\mathbb{N}^{\mathbb{N}} = \bigcup_{\xi < \omega_1} X_\xi$. For $x, y \in \mathbb{N}^{\mathbb{N}}$ define $x \preceq y$ iff $x \in X_\xi$, $y \in X_\eta$, and $\xi \leq \eta$. Thus \preceq is a prewellordering of length exactly ω_1 . It is a routine exercise to check that \preceq belongs to Σ_1^1 . \longleftarrow
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We can even slightly change the definition of \preceq to obtain a true linear order. Define $x \preceq' y$ iff either $x \in X_\xi$, $y \in X_\eta$, and $\xi < \eta$, or $x, y \in X_\xi$ for one and the same ξ and $x < y$ in the sense of the lexicographical linear order on $\mathbb{N}^{\mathbb{N}}$. Clearly \preceq' is a linear order of cofinality ω_1 and class Σ_1^1 . \square

Yet there is a rather representative class of **ROD** (that is, real-ordinal definable) linear orderings which are consistently countably cofinal. This is the subject of the next theorem.

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Theorem 2. *The following sentence is true in the Solovay model: if \leq is a Borel partial quasi-order on a (Borel) set $D \subseteq \mathbb{N}^{\mathbb{N}}$, $X \subseteq D$ is a **ROD** set, and $\leq \restriction X$ is a linear quasi-order, then $\leq \restriction X$ is countably cofinal.* $\xleftarrow{\text{m}}$

A *partial quasi-order*, PQO for brevity, is a binary relation \leq satisfying $x \leq y \wedge y \leq z \implies x \leq z$ and $x \leq x$ on its domain. In this case, an *associated equivalence relation* \equiv and an *associated strict partial order* $<$ are defined so that $x \equiv y$ iff $x \leq y \wedge y \leq x$, and $x < y$ iff $x \leq y \wedge y \not\leq x$. A PQO is *linear*, LQO for brevity, if we have $x \leq y \vee y \leq x$ for all x, y in its domain.

A PQO $\langle X; \leq \rangle$ (meaning: X is the domain of \leq) is *Borel* iff the set X is a Borel set in a suitable Polish space \mathbb{X} , and the relation \leq (as a set of pairs) is a Borel subset of $\mathbb{X} \times \mathbb{X}$.

Thus it is consistent with **ZFC** that **ROD** linear suborders of Borel PQOs are necessarily countably cofinal. Accordingly it is consistent with **ZF** + **DC** that any linear suborders of Borel PQOs are countably cofinal.

By the *Solovay model* we understand a model of **ZFC** in which all **ROD** sets of reals have some basic regularity properties, for instance, are Lebesgue measurable, have the Baire property, see [6]. We'll make use of the following two results related to the Solovay model.

Proposition 3 (Stern [7]). *It holds in the Solovay model that if $\rho < \omega_1$ then there is no **ROD** ω_1 -sequence of pairwise different sets in Σ_ρ^0 .* $\xleftarrow{\text{p2}}$ \square

Proposition 4. *It holds in the Solovay model that if \leq is a **ROD** LQO on a set $D \subseteq \mathbb{N}^{\mathbb{N}}$ then there exist a **ROD** antichain $A \subseteq 2^{<\omega_1}$ and a **ROD** map $\vartheta : D \longrightarrow A$ such that $x \leq y \iff \vartheta(x) \leq_{\text{lex}} \vartheta(y)$ for all $x, y \in D$.* $\xleftarrow{\text{p3}}$ \square

A few words on the notation. The set $2^{<\omega_1} = \bigcup_{\xi < \omega_1} 2^\xi$ consists of all transfinite binary sequences of length $< \omega_1$, and if $\xi < \omega_1$ then 2^ξ is the set of all binary sequences of length exactly ξ . A set $A \subseteq 2^{<\omega_1}$ is an *antichain* if we have $s \not\subset t$ for any $s, t \in A$, where $s \subset t$ means that t is a proper extension of s . By \leq_{lex} we denote the lexicographical order on $2^{<\omega_1}$, that is, if $s, t \in 2^{<\omega_1}$ then $s \leq_{\text{lex}} t$ iff either 1) $s = t$ or 2) $s \not\subset t$, $t \not\subset s$, and the least ordinal $\xi < \text{dom } s, \text{dom } t$ such that $s(\xi) \neq t(\xi)$ satisfies $s(\xi) < t(\xi)$. Obviously \leq_{lex} linearly orders any antichain $A \subseteq 2^{<\omega_1}$.

Proposition 4 follows from Theorem 6 in [5] saying that if, in the Solovay model, \leq is a **ROD** PQO on a set $D \subseteq \mathbb{N}^{\mathbb{N}}$ then:

either a condition (I^s) holds, which for LQO relations \leq is equivalent to the existence of A and ϑ as in Proposition 4,

or a condition (II) holds, which is incompatible with \leq being a LQO.

Thus we obtain Proposition 4 as an immediate corollary.

The next simple fact will be used below.

Lemma 5. *If $\xi < \omega_1$ then any set $C \subseteq 2^\xi$ is countably \leq_{lex} -cofinal, that is, there is a set $C' \subseteq C$, at most countable and \leq_{lex} -cofinal in C . \square* \leftarrow_{cc}

Proof (Theorem 2). *We argue in the Solovay model.* Suppose that \leq is a Borel PQO on a (Borel) set $D \subseteq \mathbb{N}^\mathbb{N}$, $X \subseteq D$ is a **ROD** set, and $\leq \upharpoonright X$ is a LQO. Our goal will be to show that $\leq \upharpoonright X$ is countably cofinal, that is, there is a set $Y \subseteq X$, at most countable and \leq -cofinal in X .

The restricted order $\leq \upharpoonright X$ is **ROD**, of course, and hence, by Proposition 4, there is a **ROD** map $\vartheta : X \rightarrow A$ onto an antichain $A \subseteq 2^{<\omega_1}$ (also obviously a **ROD** set) such that $x \leq y \iff \vartheta(x) \leq_{\text{lex}} \vartheta(y)$ for all $x, y \in X$.

If $\xi < \omega_1$ then let $A_\xi = A \cap 2^\xi$ and $X_\xi = \{x \in D : \vartheta(x) \in A_\xi\}$.

Case 1: there is an ordinal $\xi_0 < \omega_1$ such that A_{ξ_0} is \leq_{lex} -cofinal in A . However, by Lemma 5, there is a set $A' \subseteq A_{\xi_0}$, at most countable and \leq_{lex} -cofinal in A_{ξ_0} , and hence \leq_{lex} -cofinal in A as well by the choice of ξ_0 . If $s \in A'$ then pick an element $x_s \in X$ such that $\vartheta(x_s) = s$. Then the set $Y = \{x_s : s \in A'\}$ is a countable subset of X , \leq -cofinal in X , as required.

Case 2: not Case 1. That is, for any $\eta < \omega_1$ there is an ordinal $\xi < \omega_1$ and an element $s \in A_\xi$ such that $\eta < \xi$ and $t <_{\text{lex}} s$ for all $t \in A_\eta$. Then the sequence of sets

$$D_\xi = \{z \in D : \exists x \in X (z \leq x \wedge \vartheta(x) \in A_\xi)\}$$

is **ROD** and has uncountably many pairwise different terms.

We are going to get a contradiction. Recall that \leq is a Borel relation, hence it belongs to Σ_ρ^0 for an ordinal $1 \leq \rho < \omega_1$. Now the goal is to prove that all sets D_ξ belong to Σ_ρ^0 as well — this contradicts to Proposition 3, and the contradiction accomplishes the proof of the theorem.

Consider an arbitrary ordinal $\xi < \omega_1$. By Lemma 5 there exists a countable set $A' = \{s_n : n < \omega\} \subseteq A_\xi$, \leq_{lex} -cofinal in A_ξ . If $n < \omega$ then pick an element $x_n \in X$ such that $\vartheta(x_n) = s_n$. Note that by the choice of ϑ any other element $x \in X$ with $\vartheta(x) = s_n$ satisfies $x \equiv x_n$, where \equiv is the equivalence relation on D associated with \leq . It follows that

$$D_\xi = \bigcup_n X_n, \quad \text{where} \quad X_n = \{z \in D : z \leq x_n\},$$

so each X_n is a Σ_ρ^0 set together with \leq , and so is D_ξ as a countable union of sets in Σ_ρ^0 .

\square (Theorem 2)

We continue with a few remarks and questions.

Problem 6. Can one strengthen Theorem 2 as follows: *the restricted relation $\leq \restriction X$ has no monotone ω_1 -sequences?* Lemma 5 admits such a strengthening: if $\xi < \omega_1$ then easily any \leq_{lex} -monotone sequence in 2^ξ is countable. \square

Using Shoenfield's absoluteness, we obtain:

Corollary 7. *If \leq is a Borel PQO on a (Borel) set $D \subseteq \mathbb{N}^\mathbb{N}$, $X \subseteq D$ is a Σ_1^1 set, and $\leq \restriction X$ is a linear quasi-order, then $\leq \restriction X$ is countably cofinal.* $\xleftarrow{\text{mc}}$

Note that Corollary 7 fails for arbitrary LQOs of class Σ_1^1 (that is, not necessarily linear suborders of Borel PQOs), see Example 1.

Proof. In the case considered, the property of countable cofinality of $\leq \restriction X$ can be expressed by a Σ_2^1 formula. Thus it remains to consider a Solovay-type extension of the universe and refer to Theorem 2.¹ \square

Yet there is a really elementary proof of Corollary 7.

Let Y be the set of all elements $y \in D$ \leq -comparable with *every* element $x \in X$. This is a Σ_1^1 set, and $X \subseteq Y$ (as \leq is linear on X). Therefore there is a Borel set Z such that $X \subseteq Z \subseteq Y$. Now let U be the set of all $z \in Z$ \leq -comparable with *every* element $y \in Y$. Still this is a Σ_1^1 set, and $X \subseteq U$ by the definition of Y . Therefore there is a Borel set W such that $X \subseteq W \subseteq U$. And by definition still \leq is linear on W . It follows that W does not have increasing ω_1 -sequences, and hence neither does X .

Problem 8. Is Corollary 7 true for Π_1^1 sets X ?

We cannot go much higher though. Indeed, if \leq is, say, the eventual domination order on $\mathbb{N}^\mathbb{N}$, then the axiom of constructibility implies the existence of a \leq -monotone ω_1 -sequence of class Δ_2^1 . \square

Now a few words on Borel PQOs \leq having the following property:

- (*) if X is a countable set in the domain of \leq then there is an element y such that $x < y$ (in the sense of the corresponding strict ordering) for all $x \in X$.

A thoroughful study of some orderings of this type (for instance, the ordering on \mathbb{R}^ω defined so that $x \leq y$ iff either $x(n) = y(n)$ for all but finite n or $x(n) < y(n)$ for all but finite n) was undertaken in early papers of Felix

¹ We'll not discuss the issue of an inaccessible cardinal on the background.

Hausdorff, e.g., [2, 3] (translated to English in [4]). In particular, Hausdorff investigated the structure of *pantachies*, that is, maximal linearly ordered subsets of those partial orderings. As one of the first explicit applications of the axiom of choice, Hausdorff established the existence of a pantachy in any partial order, and made clear distinction between such an existence proof and an actual, well-defined construction of an individual pantachy (see [2], p. 110). The next result shows that the latter is hardly possible in **ZFC**, at least if we take for granted that any individual set-theoretic construction results in a **ROD** set.

Corollary 9. *The following sentence is true in the Solovay model: if \leq is a Borel partial quasi-order on a (Borel) set $D \subseteq \mathbb{N}^{\mathbb{N}}$, satisfying (*), then \leq has no **ROD** pantachies.* \dashv

Proof. It follows from (*) that any pantachy in $\langle D; \leq \rangle$ is a set of uncountable cofinality. Now apply Theorem 2. \square

A further corollary: it is impossible to prove the existence of pantachies in any Borel PQO satisfying (*) in **ZF** + **DC**.

References

- [1] L. A. Harrington, D. Marker, and S. Shelah, Borel orderings. *Trans. Amer. Math. Soc.*, 1988, 310, pp. 293–302.
- [2] F. Hausdorff, Untersuchungen über Ordnungstypen IV, V. *Ber. über die Verhandlungen der Königlich Sächsische Gesellschaft der Wissenschaften zu Leipzig, Math.-phys. Klasse*, 1907, 59, pp. 84–159.
- [3] F. Hausdorff, Die Graduierung nach dem Endverlauf. *Abhandlungen der Königlich Sächsische Gesellschaft der Wissenschaften zu Leipzig, Math.-phys. Klasse*, 1909, 31, pp. 295–334.
- [4] F. Hausdorff, *Hausdorff on ordered sets*, Translated from the German, edited and with commentary by J. M. Plotkin. History of Mathematics, 25. AMS, Providence, RI and LMS, London, 2005. xviii+322 pp.
- [5] Vladimir Kanovei. Linearization of definable order relations. *Annals of Pure and Applied Logic*, 2000, 102, 1-2, pp. 69–100.
- [6] R.M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. Math. (2)*, 1970, 92, pp. 1–56.
- [7] J. Stern. On Lusin’s restricted continuum problem. *Ann. Math. (2)*, 1984, 120, pp. 7–37.